

Star-critical Ramsey number of K_4 versus F_n

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Abstract

For two graphs G and H , the Ramsey number $r(G, H)$ is the smallest positive integer r , such that any red/blue coloring of the edges of graph K_r contains either a red subgraph that is isomorphic to G or a blue subgraph that is isomorphic to H . Let $S_k = K_{1,k}$ be a star of order $k + 1$ and $K_n \sqcup S_k$ be a graph obtained from K_n by adding a new vertex v and joining v to k vertices of K_n . The star-critical Ramsey number $r_*(G, H)$ is the smallest positive integer k such that any red/blue coloring of the edges of graph $K_{r-1} \sqcup S_k$ contains either a red subgraph that is isomorphic to G or a blue subgraph that is isomorphic to H , where $r = r(G, H)$. In this paper, it is shown that $r_*(F_n, K_4) = 4n + 2$, where $n \geq 4$.

Keywords: Ramsey number, Star-critical, Fan, Complete graph.

MSC: 05C55; 05D10

1. Introduction and Background

Let $G = (V(G), E(G))$ denote a finite simple graph on the vertex set $V(G)$ and the edge set $E(G)$. The *order* of a graph G is $|V(G)|$. The subgraph of G *induced* by $S \subseteq V(G)$, $G[S]$, is a graph with vertex set S and two vertices of S are adjacent in $G[S]$ if and only if they are adjacent in G . For a vertex $v \in V(G)$, we denote the set of all neighbors of v by $N(v)$. For the subset $S \subseteq V(G)$, $N(S) = \bigcup_{s \in S} N(s)$. The degree of a vertex v in G is denoted by $d_G(v)$ (for abbreviation $d(v)$). The graph $G - H$ is the subgraph

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of G obtaining from the deletion of the vertices of H where H is a subgraph of G .

The cycle of order n is denoted by C_n . We refer to a cycle of odd order as an odd cycle. We denote the complete graph on r vertices by K_r . By *triangle*, we refer to the complete graph K_3 . A *clique* is a subset of vertices of a graph, such that its induced subgraph is complete. The number of vertices of a largest clique in a graph G is denoted by $\omega(G)$. The *fan* graph, F_n , can be constructed by joining n copies of the complete graph K_3 with a common vertex. We refer to this common vertex as the *center* vertex of the fan graph. For any graph G and positive integer n , the disjoint union of n copies of G is denoted by nG .

For a positive integer k , a complete k -partite graph is a graph that can be partitioned into k disjoint sets, such that no two vertices within the same set are adjacent, but every pair of vertices from two different sets are adjacent. Now, let G be a complete k -partite graph with disjoint sets A_1, \dots, A_k which we sometimes write $G = (A_1, \dots, A_k)$, then the graph G is denoted by $K_{|A_1|, \dots, |A_k|}$.

For a red/blue edge-coloring of a graph G , the subgraph induced by red edges is denoted by G^R and the subgraph induced by blue edges is denoted by G^B . we denote the set of all neighbors of v in G^R by $N^R(v)$ and in G^B by $N^B(v)$. We sometimes refer to a vertex $u \in N^B(v)$ (or $u \in N^R(v)$) as a blue neighbor (or red neighbor) of v . The number of all blue neighbors of v in G is denoted by $d^B(v)$. In the other words, $d^B(v) = |N^B(v)|$. We define $d^R(v)$ similarly. Let H be a subgraph of G or a subset of $V(G)$ and $v \in V(G)$. The set $N_H(v)$ is the set of neighbors of v in H . In the other words, $N_H(v) = N(v) \cap H$. Also, we define the sets $N_H^B(v) = N^B(v) \cap H$ and $N_H^R(v) = N^R(v) \cap H$. The $d_H^B(v)$ and $d_H^R(v)$ are defined similarly.

Let $S_k = K_{1,k}$ be a *star* of order $k+1$ and $K_n \sqcup S_k$ be a graph obtained from K_n by adding a new vertex v and joining v to k vertices of K_n .

Let G and H be two graphs. The *Ramsey number* of G and H is the smallest positive integer r such that every red/blue coloring of K_r contains a red G or a blue H . Note that for $r = r(G, H)$, there exists a critical red/blue edge-coloring graph K_{r-1} which contains no monochromatic subgraph isomorphic to G or H . We call such a red/blue edge-coloring a (G, H) -free coloring and the complete graph, K_{r-1} , with a (G, H) -free coloring as a (G, H) -free graph. For two graphs G and H , the *star-critical Ramsey number*, $r_*(G, H)$, is defined to be the smallest integer k such that every red/blue edge-coloring of $K_{r-1} \sqcup S_k$ contains a red G or a blue H , where $r = r(G, H)$.

The star-critical Ramsey number was defined by Hook and Isaak in [1]. The following theorem was Shown by Zhen Li and Yusheng Li in [2].

Theorem 1. *Let $n \geq 2$ be an integer. Then $r_*(F_n, K_3) = 2n + 2$.*

In this paper, we prove the following theorem.

Theorem 2. *Let $n \geq 4$ be an integer. Then $r_*(F_n, K_4) = 4n + 2$.*

2. Proof of Theorem 2

In order to prove our theorem, we need some theorems and lemmas.

Lemma 3. [3] *Let $m \geq 1$ and $n \geq 2$ be integers. Then $r(mK_2, K_n) = n + 2m - 2$. In particular, $r(nK_2, K_4) = 2n + 2$.*

Theorem 4. [4] *$r(F_n, K_3) = 4n + 1$ for $n \geq 2$.*

Theorem 5. [5] *Let $n \geq 2$ be an integer. Then $r(F_n, K_4) = 6n + 1$.*

Proposition 6. *Let $G = K_{6n}$ be a (F_n, K_4) -free graph, where $n \geq 2$. For each $v \in V(G)$, $2n - 1 \leq d^R(v) \leq 2n + 1$. Consequently, $4n - 2 \leq d^B(v) \leq 4n$.*

Proof. If $d^R(v) \geq 2n + 2$, then by Lemma 3, $G[N^R(v)]$ contains a red nK_2 or a blue K_4 . If $d^R(v) \leq 2n - 2$, then $d^B(v) \geq 4n + 1$. By Theorem 4, $G[N^B(v)]$ contains a blue K_3 or a red F_n . In both cases, we reach to a contradiction. \square

Proposition 7. *Let $n \geq 2$ and G be a (F_n, K_4) -free graph of order $6n$ which contains a red K_{2n} , say K . Then $G - K$ contains no blue triangle.*

Proof. By contrary, suppose that $G - K$ contains a blue triangle, say $T = xyz$. If $d_K^R(u) \geq 2$, for some $u \in \{x, y, z\}$, then $K \cup \{u\}$ contains a red F_n , which is a contradiction. Therefore, $d_K^B(u) \geq 2n - 1$, for every $u \in V(T)$. Thus, $N^B(x) \cap N^B(y) \cap N^B(z) \neq \emptyset$, which implies that G contains a blue K_4 , a contradiction. \square

Proposition 8. *Let $n \geq 4$ and G be a (F_n, K_4) -free graph of order $6n$ which contains a red K_{2n} , say K . Also, suppose that C_t is a shortest blue odd cycle in $G - K$. Then $t = 5$ or $t = 7$.*

Proof. Proposition 7 implies that $t \geq 5$. Let u be an arbitrary vertex of C_t . By Proposition 6, since $d^B(u) \geq 4n - 2$, we conclude that $d_{G-(K \cup C_t)}^B(u) \geq 2n - 4$. Since $G - K$ contains no blue triangle, each two adjacent vertices of C_t , say u and v , have at least $4n - 8$ distinct blue neighbors in $G - (K \cup C_t)$. Thus, $4n - t \geq 4n - 8$, which implies that $t \leq 8$ and hence $t = 5$ or $t = 7$. \square

Proposition 9. *Let $n \geq 4$ and G be a (F_n, K_4) -free graph of order $6n$ which contains a red K_{2n} , say K . Then $G - K$ contains no blue C_5 .*

Proof. By contrary, suppose that G contains a blue C_5 . Note that Proposition 7 implies that this cycle is an induced cycle in G^B . Let $V(C_5) = \{u_1, \dots, u_5\}$ such that u_i is adjacent to u_{i+1} , for $i = 1, \dots, 4$ and u_5 is adjacent to u_1 . For each $u_i \in V(C_5)$, we have $|N_{G-(K \cup C_5)}^B(u_i)| \geq 2n - 4$.

Let $X_i \subseteq N_{G-(K \cup C_5)}^B(u_i)$ with $|X_i| = 2n - 4$, for $i = 1, 2, \dots, 5$. Then Proposition 7 implies that $R_1 = X_1 \cup \{u_2, u_5\}$ and $R_2 = X_2 \cup \{u_1, u_3\}$ are disjoint red cliques of order $2n - 2$ (See Figure 1). Let a, b and c be the remaining vertices of G . Proposition 7 implies that at least one of the edges between a, b, c is red, for instance ab .

Suppose that all edges between two sets $\{a, b, c\}$ and $\{u_1, u_2\}$ are red. Then all edges between c and $R_1 \cup R_2 - \{u_1, u_2\}$ are blue, otherwise $R_1 \cup \{a, b, c\}$ or $R_2 \cup \{a, b, c\}$ contains a red F_n with the center u_2 or u_1 , respectively. So, $d_{G-K}^B(c) \geq 4n - 6 = (2n - 2) + (2n - 2) - 2$ and consequently $d_K^B(c) \leq 6$, since $d^B(c) \leq 4n$. Now, since $n \geq 4$, $K \cup \{c\}$ contains a red F_n , which is a contradiction.

So, with no loss of generality, we may assume that a has a blue edge to u_1 . Proposition 7 implies that all edges between a and R_1 are red. Let $R'_1 = R_1 \cup \{a\}$. By Proposition 7, at least one of the edges between b, c, u_4 is red, for instance bc . Similarly, all edges of the set $\{b, c\}$ to the set $\{u_1, u_2\}$ can not be red. If all edges of $\{b, c\}$ to the set $\{u_1, u_2\}$ are red, then all edges between u_4 and $R_1 \cup R_2 - \{u_1, u_2\}$ are blue, otherwise $R_1 \cup \{b, c, u_4\}$ or $R_2 \cup \{b, c, u_4\}$ contains a red F_n with the center u_2 or u_1 , respectively. It means $d_{G-K}^B(u_4) \geq 4n - 6$ and consequently, $d_K^B(u_4) \leq 6$. Since $n \geq 4$, $K \cup \{u_4\}$ contains a red F_n , which is a contradiction.

With no loss of generality, suppose that bu_1 or bu_2 is blue (Note that this case for c is similar).

If bu_1 is blue, then by Proposition 7 all edges between b and R'_1 are red. So, $R''_1 = R'_1 \cup \{b\}$ is a red clique of order $2n$. Since G contains no red F_n , $d_{R''_1}^R(u_3), d_{R''_1}^R(u_4) \leq 1$. So, u_3 and u_4 have at least $2n - 2$ blue common neighbors in R''_1 , a contradiction with the fact that $G - K$ has no blue triangle.

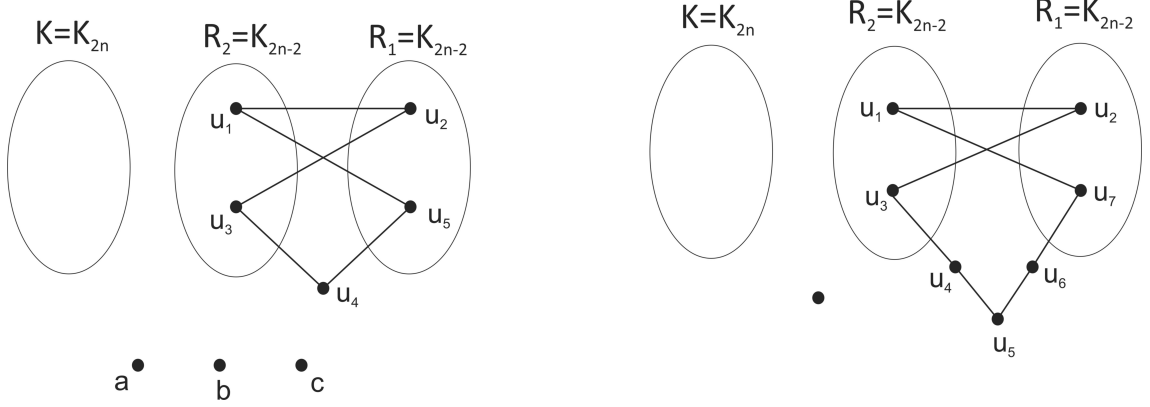


Figure 1: $G - K$ containing an induced blue C_5 or C_7 .

If bu_2 is blue, then all the edges between b and R_2 are red, by Proposition 7. Then $R'_2 = R_2 \cup \{b\}$ is a red clique of order $2n - 1$. By Proposition 7, one of the edges cu_1 or cu_2 is red, for instance cu_1 . The edge cu_4 is blue, otherwise $R'_2 \cup \{c, u_4\}$ contains a red F_n with the center u_1 . Thus, cu_3 and cu_5 are red since $G - K$ contains no blue K_3 . Therefore $R'_2 \cup \{c, u_5\}$ contains a red F_n with the center u_3 , a contradiction. \square

Proposition 10. *Let $n \geq 4$ and G be a (F_n, K_4) -free graph of order $6n$ which contains a red K_{2n} , say K . Then $G - K$ contains no blue C_7 .*

Proof. By contrary, suppose that G contains a blue C_7 . Propositions 7 and 9 yield that this cycle is an induced cycle in G^B . Let $V(C_7) = \{u_1, \dots, u_7\}$ such that u_i is adjacent to u_{i+1} , for $i = 1, \dots, 6$ and u_7 is adjacent to u_1 . By Proposition 6, for every $u_i \in V(C_7)$, we have $d_{G-(K \cup C_7)}^B(u_i) \geq 2n - 4$. Let $X_i \subseteq N_{G-(K \cup C_7)}^B(u_i)$, for $i = 1, 2, \dots, 7$. Then $R_1 = X_1 \cup \{u_2, u_7\}$ and $R_2 = X_2 \cup \{u_1, u_3\}$ are red cliques of order $2n - 2$ since $G - K$ contains no blue triangle (See Figure 1). It is easy to see that $R_2 \cup \{u_4, u_5, u_6\}$ induces a red F_n with the center u_1 , a contradiction. \square

Now, we can prove the following.

Lemma 11. *Let $n \geq 4$ be an integer and G^R be the graph induced by red edges from a (F_n, K_4) -free coloring of $G = K_{6n}$. If G^R contains K_{2n} , then it contains $3K_{2n}$.*

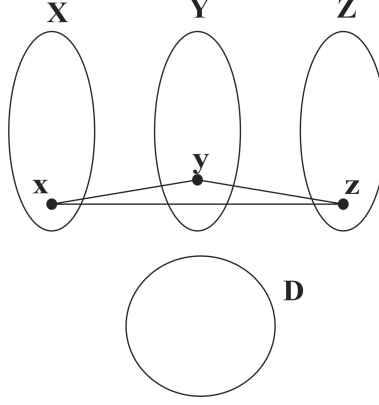


Figure 2: The graph $G = K_{6n}$ containing a blue triangle T .

Proof. By Propositions 7, 9 and 10, we can conclude that $G - K$ contains no blue odd cycle. Thus, $G^B[G - K]$ is bipartite. Since G contains no red F_n , the size of each part is at most $2n$ and hence there exists a $3K_{2n}$ in G^R . \square

Lemma 12. *Suppose $G = K_{6n}$ is a (F_n, K_4) -free graph, where $n \geq 4$. Then $\omega(G^R) = 2n$.*

Proof. Let $G = K_{6n}$ be a (F_n, K_4) -free graph such that $\omega(G^R) \leq 2n - 1$. Since $R(F_n, K_3) = 4n + 1$, G has a blue K_3 , say $T = xyz$. Since $d_{G^B}(s) \geq 4n - 2$, for all $s \in V(G)$, we conclude that $d_{G-T}^B(v) \geq 4n - 4$, for all $v \in V(T)$. Note that $|N^B(u) \cap N^B(v)| \geq 2n - 5 = (4n - 4) + (4n - 4) - (6n - 3)$, for $u, v \in V(T)$.

Let $N_{xy} = N^B(x) \cap N^B(y)$. Define N_{xz} and N_{yz} similarly. If there exists a blue edge in N_{xy} , then G contains a blue K_4 , a contradiction. Similar argument holds for N_{yz} and N_{xz} . Therefore N_{xy} , N_{xz} and N_{yz} are red cliques of order at least $2n - 5$. Note that $N_{xy} \cap N_{xz} = N_{xy} \cap N_{yz} = N_{xz} \cap N_{yz} = \emptyset$ and all the edges between z and N_{xy} are red, otherwise G contains a blue K_4 . So, $Z = N_{xy} \cup \{z\}$ is a red clique of order at least $2n - 4$. Thus, $\omega(G^R) \geq 2n - 4$. Similarly, $X = N_{yz} \cup \{x\}$ and $Y = N_{xz} \cup \{y\}$ are the red cliques of order at least $2n - 4$. Let $X' \subseteq X$ be a red clique of order $2n - 4$ which contains x . Define Y' and Z' similarly (See Figure 2).

Let $D = V(G) - (X' \cup Y' \cup Z')$, then $|D| = 12$. Note that all edges between x and $Y' \cup Z'$ are blue and consequently $d_D^B(x) \geq 6 = 4n - 2 - (2n - 4 + 2n - 4)$. Similar argument holds for y and z . Since $|D| = 12$, there are two vertices in $V(T)$ which have at least two blue common neighbors in D . With no loss

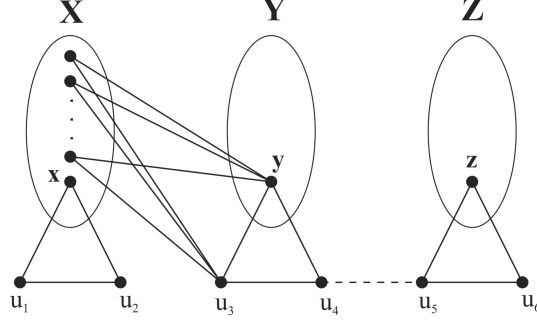


Figure 3

of generality, assume that $|N_D^B(x) \cap N_D^B(y)| \geq 2$. One can easily see that all edges between $N_D^B(x) \cap N_D^B(y)$ and Z are red since G contains no blue K_4 . Therefore $\omega(G^R) \geq 2n - 2$. So, we consider two cases:

Case 1: $\omega(G^R) = 2n - 2$.

Since $\omega(G^R) = 2n - 2$ and $|N_D^B(s)| \geq 6$, for all $s \in V(T)$, we conclude that $|N_D^B(x) \cap N_D^B(y)| = |N_D^B(x) \cap N_D^B(z)| = |N_D^B(y) \cap N_D^B(z)| = 2$.

Notice that $|X| = |Y| = |Z| = 2n - 2$. Let $D' \subseteq D$ be the remaining 6 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . Let $D' = \{u_1, \dots, u_6\}$ such that $N_{D'}^B(x) = \{u_1, u_2\}$, $N_{D'}^B(y) = \{u_3, u_4\}$ and $N_{D'}^B(z) = \{u_5, u_6\}$.

Assume that u_1u_2 is red. Then all edges between two sets $\{u_5, u_6\}$ and $Y - \{y\}$ are blue. Otherwise if there exists a red edge, for instance u_5v , where $v \in Y - \{y\}$, then $Y \cup \{u_1, u_2, u_5\}$ induces a red F_n with the center y . Now, $\{u_5, u_6, v, z\}$ induces a blue K_4 , where $v \in Y - \{y\}$.

So, we may assume that u_1u_2 , u_3u_4 and u_5u_6 are blue. Note that since G contains no blue K_4 , there exists a red edge in $G[\{u_3, u_4, u_5, u_6\}]$, say u_4u_5 . All edges between u_3 and $X - \{x\}$ are blue. Otherwise, if there exists a red edge u_3v , where $v \in X - \{x\}$, then $X \cup \{u_3, u_4, u_5\}$ induces a red F_n with the center x , a contradiction (See Figure 3).

Now, all edges between u_4 and X are red. Otherwise, if there exists a blue edge, for instance u_4v , where $v \in X - \{x\}$, then $\{u_3, u_4, v, y\}$ induces a blue K_4 , a contradiction. But now $X \cup \{u_4\}$ induces a red clique of order $2n - 1$, a contradiction.

Case 2: $\omega(G^R) = 2n - 1$.

In this case, $|N_D^B(u) \cap N_D^B(v)| = 3$, for some 2-subset $\{u, v\} \subseteq V(T)$. We

divide this case into some subcases:

Subcase 2.1: $|N_D^B(u) \cap N_D^B(v)| = 3$, for every 2-subset $\{u, v\} \subseteq V(T)$.

In this subcase we have $|X| = |Y| = |Z| = 2n - 1$. Let $D' \subseteq D$ be the remaining 3 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . The vertex $v \in D'$ has at least one blue edge to X, Y and Z since $\omega(G^R) = 2n - 1$. Suppose that uv is a blue edge such that $u \in X$. Notice that $d_{Y \cup Z}^B(v) \geq 2n - 3 = (4n - 2) - 2 - (2n - 1)$ since $d_{G^B}(v) \geq 4n - 2$. Let $L_v = N_{Y \cup Z}^B(v)$, then $|L_v| \geq 2n - 3$. Since G contains no red F_n , we conclude $d_Y^R(u), d_Z^R(u) \leq 1$. Thus, $d_{L_v}^B(u) \geq 2n - 5$. Let $L_u = N_{Y \cup Z}^B(u) \cap L_v$. Then $|L_u| \geq 2n - 5$ and L_u induces a red clique, otherwise $\{u, v\} \cup L_u$ induces a blue K_4 . Now, $Y \cup L_u$ or $Z \cup L_u$ induces a red F_n since $n \geq 4$, a contradiction.

Subcase 2.2:

$|N_D^B(x) \cap N_D^B(y)| = 3, |N_D^B(x) \cap N_D^B(z)| = 2, |N_D^B(y) \cap N_D^B(z)| = 1$.

In this subcase, we have $|X| = 2n - 3, |Y| = 2n - 2, |Z| = 2n - 1$. Let $D' \subseteq D$ be the remaining 6 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . Also, let $D' = \{u_1, u_2, \dots, u_6\}$. We have $d_{D'}^B(x) = 1, d_{D'}^B(y) = 2$ and $d_{D'}^B(z) = 3$ since $d_{G^B}(s) \geq 4n - 2$, for all $s \in V(G)$. With no loss of generality, assume that $N_{D'}^B(x) = \{u_1\}, N_{D'}^B(y) = \{u_2, u_3\}$ and $N_{D'}^B(z) = \{u_4, u_5, u_6\}$. One of the edges u_4u_5, u_5u_6 and u_4u_6 is red, otherwise $\{z, u_4, u_5, u_6\}$ induces a blue K_4 . With no loss of generality, suppose that u_5u_6 is red.

Note that $\{u_1, u_2, u_3\}$ induces a blue K_3 , otherwise there exists a red F_n with center z . Also, u_iu_4 is blue, for $i = 1, 2, 3$. If u_1u_4 is red, then $Y \cup \{u_1, u_4, u_5, u_6\}$ induces a red F_n , a contradiction. Also, if u_iu_4 is red, for $i = 2$ or $i = 3$, then $\{u_i, u_4, u_5, u_6\}$ induces a red F_n , a contradiction. Now, $\{u_1, u_2, u_3, u_4\}$ induces a blue K_4 which is a contradiction.

Subcase 2.3:

$|N_D^B(x) \cap N_D^B(y)| = |N_D^B(x) \cap N_D^B(z)| = 3, |N_D^B(y) \cap N_D^B(z)| = 0$.

In this subcase, we have $|X| = 2n - 4, |Y| = 2n - 1, |Z| = 2n - 1$. Let $D' \subseteq D$ be the remaining 6 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . Also, let $D' = \{u_1, u_2, \dots, u_6\}$. We have $d_{D'}^B(x) = 0, d_{D'}^B(y) = 3$ and $d_{D'}^B(z) = 3$ since $d_{G^B}(s) \geq 4n - 2$, for all $s \in V(G)$. With no loss of generality, suppose that $N_{D'}^B(y) = \{u_1, u_2, u_3\}$. Note that there exists a red edge u_iu_j , where $1 \leq i < j \leq 3$, otherwise $\{y, u_1, u_2, u_3\}$ induces a blue K_4 . Now, $Z \cup \{u_i, u_j\}$ induces a red F_n with the center z , a contradiction.

Subcase 2.4:

$$|N_D^B(x) \cap N_D^B(y)| = |N_D^B(x) \cap N_D^B(z)| = 3, |N_D^B(y) \cap N_D^B(z)| = 1.$$

In this subcase, we have $|X| = 2n - 3$, $|Y| = 2n - 1$, $|Z| = 2n - 1$. Let $D' \subseteq D$ be the remaining 5 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . Also, let $D' = \{u_1, u_2, \dots, u_5\}$. We have $d_{D'}^B(y) \geq 2$ and $d_{D'}^B(z) \geq 2$ since $d_{G^B}(s) \geq 4n - 2$, for all $s \in V(G)$. With no loss of generality, suppose that $\{u_1, u_2\} \subseteq N_{D'}^B(y)$ and $\{u_3, u_4\} \subseteq N_{D'}^B(z)$.

Suppose that $u_5 \in N^B(z)$. Then $G[\{u_3, u_4, u_5\}]$ has a red edge, say u_4u_5 , otherwise $\{z, u_3, u_4, u_5\}$ induces a blue K_4 . Now, $Y \cup \{u_4, u_5\}$ induces a red F_n , a contradiction. Similar argument holds for y .

So, we may assume that $u_5 \in N^R(y) \cap N^R(z)$. Note that u_4u_5 is blue, otherwise $Y \cup \{u_4, u_5\}$ induces a red F_n with the center y . Similarly, u_iu_5 is blue, for $i = 1, 2, 3$.

u_1u_2 is blue, otherwise $Z \cup \{u_1, u_2\}$ induces a red F_n with the center z . Similarly, u_3u_4 is blue. Since G contains no blue K_4 , we may assume that u_2u_3 is red. Then u_1u_4 is blue, otherwise $X \cup \{u_1, u_2, u_3, u_4\}$ induces a red F_n . Also, note that u_1u_3 or u_2u_4 is blue, otherwise $X \cup \{u_1, u_2, u_3, u_4\}$ induces a red F_n . Suppose that u_1u_3 is blue. Now, $\{u_1, u_3, u_4, u_5\}$ induces a blue K_4 , a contradiction.

Subcase 2.5:

$$|N_D^B(x) \cap N_D^B(y)| = |N_D^B(x) \cap N_D^B(z)| = 3, |N_D^B(y) \cap N_D^B(z)| = 2.$$

In this subcase, we have $|X| = 2n - 2$, $|Y| = 2n - 1$, $|Z| = 2n - 1$. Let $D' \subseteq D$ be the remaining 4 vertices which each one of them is not the common neighbor of any two vertices of x, y and z in G^B . Also, let $D' = \{u_1, u_2, u_3, u_4\}$. We have $d_{D'}^B(x) \geq 0$, $d_{D'}^B(y) \geq 1$ and $d_{D'}^B(z) \geq 1$. With no loss of generality, suppose that $u_1 \in N_{D'}^B(y)$ and $u_2 \in N_{D'}^B(z)$. The vertex u_1 has at least one blue edge to Z since $\omega(G^R) = 2n - 1$.

Suppose that u_1 has a blue edge to X . By the fact that $d_{G^B}(u_1) \geq 4n - 2$, the vertex u_1 has at least $2n - 4 = 4n - 2 - (2n - 1 + 3)$ blue edges to $X \cup Z$. Let $L_{u_1} = N_{X \cup Z}^B(u_1)$. Then $|L_{u_1}| \geq 2n - 4$ and L_{u_1} induces a red clique, otherwise $\{u_1, y\} \cup L_{u_1}$ induces a blue K_4 . Now, $X \cup L_{u_1}$ or $Z \cup L_{u_1}$ induces a red F_n since $n \geq 4$, a contradiction.

Now, suppose that all edges between u_1 and X are red. Let $X' = X \cup \{u_1\}$. Clearly, X' is a red clique of order $2n - 1$. Since $\omega(G^R) = 2n - 1$, u_2 has some blue edges to X' and to Y . The vertex u_2 has at least $2n - 3 = 4n - 2 - (2n - 1 + 2)$ blue edges to $X' \cup Y$. Let $L_{u_2} = N_{X' \cup Y}^B(u_2)$. Then

$|L_{u_2}| \geq 2n - 3$ and L_{u_2} induces a red clique, otherwise $\{z, u_2\} \cup L_{u_2}$ induces a blue K_4 , which is a contradiction. But, $X' \cup L_{u_2}$ or $Y \cup L_{u_2}$ induces a red F_n since $n \geq 4$, a contradiction. \square

Definition 13. Let $n \geq 2$ be an integer and $r = r(F_n, K_4) = 6n + 1$. Define a class \mathfrak{g} of graphs as the family of G_1 and G_2 . Also, every graph in \mathfrak{g} shows a red/blue edge-coloring of K_{r-1} as

$$G_1 : G_1^R = 3K_{2n} \quad , \quad G_1^B = K_{2n, 2n, 2n}$$

$$G_2 : G_2^R = 3K_{2n} \cup (I_1 \cup I_2 \cup I_3) \quad , \quad G_2^B = K_{2n, 2n, 2n} - (I_1 \cup I_2 \cup I_3)$$

such that if $K_{2n, 2n, 2n} = (A_1, A_2, A_3)$, then I_i is a collection of k_i independent edges between A_i and A_{i+1} , for $i = 1, 2$ and I_3 is a collection of k_3 independent edges between A_3 and A_1 , where $1 \leq k_i \leq 2n$, for $i = 1, 2, 3$ and $G_2[I_1 \cup I_2 \cup I_3]$ contains no red triangle.

Proof of the Theorem 2: Let $G = K_{6n}$ be a (F_n, K_4) -free graph. By Lemma 12, G contains a red K_{2n} . Thus, by Lemma 11, G contains a red $3K_{2n}$. One can easily find out that these two kind of colorings in the Definition 13 are the only (F_n, K_4) -free colorings when G contains a red $3K_{2n}$.

We denote these red K_{2n} s by G_i , for $i = 1, 2, 3$. Let w be a new vertex and G' be a graph obtaining from G by adding w and some colored edges between w and G . We want to find an upper bound for the most number of colored edges between w and G such that G' still is a (F_n, K_4) -free graph. It is clear that w has at most one red edge to G_i , for $i = 1, 2, 3$, since G' has no red F_n .

Suppose that $d_{G_i}^B(w) \geq 1$, $d_{G_j}^B(w) \geq 2$ and $d_{G_k}^B(w) \geq 3$, where $\{i, j, k\} = \{1, 2, 3\}$. With no loss of generality, let $\{u_1\} \subseteq N_{G_i}^B(w)$, $\{u_2, u_3\} \subseteq N_{G_j}^B(w)$ and $\{u_4, u_5, u_6\} \subseteq N_{G_k}^B(w)$. Note that $d_{G_j}^R(u_1) \leq 1$ since G contains no red F_n . So, we may assume that u_1u_2 is a blue edge. With the similar argument and with no loss of generality, suppose that u_2u_4 and u_2u_5 are blue edges. Now, $\{u_1, u_2, u_4, u_5\}$ contains a blue triangle. Hence G' contains a blue K_4 , a contradiction.

Now, suppose that $d_{G_i}^B(w) \geq 2$, for $i = 1, 2, 3$. With no loss of generality, let $\{u_1, u_2\} \subseteq N_{G_1}^B(w)$, $\{u_3, u_4\} \subseteq N_{G_2}^B(w)$ and $\{u_5, u_6\} \subseteq N_{G_3}^B(w)$. One of the edges u_1u_5 or u_1u_6 is blue, say u_1u_5 . If u_1u_3 and u_1u_4 are both blue, then u_3u_5 and u_4u_5 are red, otherwise G' contains a blue K_4 . But now, G contains a red F_n with the center u_5 , a contradiction. So, we may assume that u_1u_3 , u_2u_4 , u_3u_5 and u_4u_6 are red. Since G contains no red F_n , u_1u_5 is

blue. Note that u_1u_4 and u_4u_5 are blue. But now, $\{u_1, u_4, u_5, w\}$ induces a blue K_4 , which is a contradiction.

Now, we consider two cases:

Case1: $d_{G_i}^B(w) \geq 1$, for $i = 1, 2, 3$.

With no loss of generality, according to what is mentioned above, assume that $d_{G_1}^B(w) = 1$. If $d_{G_2}^B(w) = 2$, then $d_{G_3}^B(w) \leq 2$ and hence $d(w) \leq 8$ (Since $d^R(w) \leq 3$). Now, if $d_{G_i}^B(w) = 1$, for $i \in \{2, 3\}$, then $d(w) \leq 2n + 4$ (Since $d_{G_1 \cup G_2}^R(w) \leq 2$). Hence in this case we have $d(w) \leq 2n + 4$.

Case2: $d_{G_i}^B(w) = 0$, for some $i \in \{1, 2, 3\}$.

Let $i = 1$. Since $d_{G_1}^R(w) \leq 1$, one can easily obtain that $d(w) \leq 4n + 1$ (at most $4n$ edges to $G_2 \cup G_3$ and at most one red edge to G_1).

These two cases imply that if $d(w) \geq 4n + 2$, then G' contains a red F_n or a blue K_4 . Hence $r_*(F_n, K_4) \leq 4n + 2$.

One can easily find out that if w has degree $4n + 1$ such that $4n$ blue edges are between w and $G_1 \cup G_2$ and one red edge is between w and G_3 , then G' is a (F_n, K_4) -free graph. Therefore, $r_*(F_n, K_4) \geq 4n + 2$, and consequently $r_*(F_n, K_4) = 4n + 2$.

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